Derivative pricing in fractional SABR model

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Outline

- Quick review on the SABR model and the SABR formula
- Lognormal fractional SABR (fSABR) model
  - A bridge representation for probability density of lognormal fSABR
  - Small time approximations of option premium and implied volatility in lognormal fSABR framework
  - Heuristic sample path large deviation principle
- Target volatility option (TVOs) pricing in lognormal fSABR
  - Decomposition formula
  - Approximations of the price of a TV call
- Conclusion

Stochastic $\alpha\beta\rho$ (SABR) model

Stochastic $\alpha\beta\rho$ (SABR) model was suggested and investigated by Hagan-Lesniewski-Woodward as

$$
\begin{aligned}
dS_t &= S_t^{\beta}\alpha_t(\rho dB_t + \bar{\rho} dW_t), \quad S_0 = s; \\
d\alpha_t &= \nu\alpha_t dB_t, \quad \alpha_0 = \alpha
\end{aligned}
$$

where B_t and W_t are independent Brownian motions, $\bar{\rho} = \sqrt{1-\rho^2}$.

- SABR model is market standard for quoting cap and swaption volatilities using the SABR formula for implied volatility. Nowadays also used in FX and equity markets.
- $\beta = 0$ is referred to as *normal SABR*
- $\beta = 1$ is referred to as *lognormal SABR*

SABR formula

The SABR formula is a small time asymptotic expansion up to first order for the implied volatilities of call/put option induced by the SABR model.

$$\sigma_{BS}(K,\tau) = \nu \frac{\log(s/K)}{D(\zeta)} \{1 + O(\tau)\}$$

as the time to expiry τ approaches 0. D and ζ are defined respectively as

$$D(\zeta) = \log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}+\zeta-\rho}{1-\rho}\right)$$

and

$$\zeta = \begin{cases} \frac{\nu}{\alpha}\frac{s^{1-\beta}-K^{1-\beta}}{1-\beta} & \text{if } \beta \neq 1; \\ \frac{\nu}{\alpha}\log\left(\frac{s}{K}\right) & \text{if } \beta = 1. \end{cases}$$

SABR formula - zeroth order

The zeroth order SABR formula is obtained by matching the exponents

$$e^{-\frac{d_*^2(s_0,\alpha_0)}{2T}} \approx C(K,T) = C_{BS}(K,T) \approx e^{-\frac{(\log s_0 - \log K)^2}{2\sigma_{BS}^2 T}}$$

thus,

$$\sigma_{BS}(K,T) \approx \frac{|\log s_0 - \log K|}{d_*(s_0,\alpha_0)}.$$

where d_* is the minimal distance from the initial point (s_0, α_0) to the half plane $\{(s,\alpha) : s \geq K\}$.

Why fractional process?

Gatheral-Jaisson-Rosenbaum observed from empirical data that

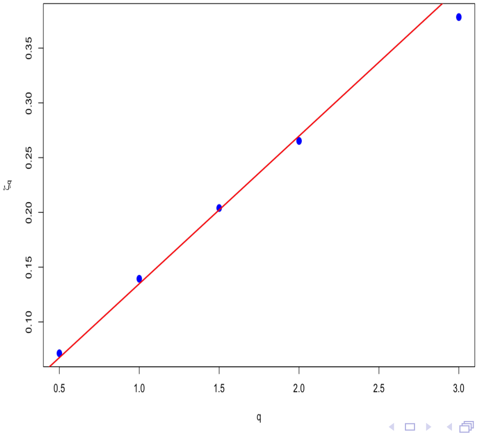
- Log-volatility behaves as a fractional Brownian Motion with Hurst exponent H of order 0.1 at any reasonable time scale. Indeed, they fitted the empirical qth moments $m(q, \Delta)$ in various lags Δ to

$$\mathbb{E}\left[|\log \sigma_{t+\Delta} - \log \sigma_t|^q\right] = K_q \Delta^{\zeta_q}$$

  proxied by daily realized variance estimates. K_q denotes the qth moment of standard normal.
- At-the-money volatility skew is well approximated by a power law function of time to expiry

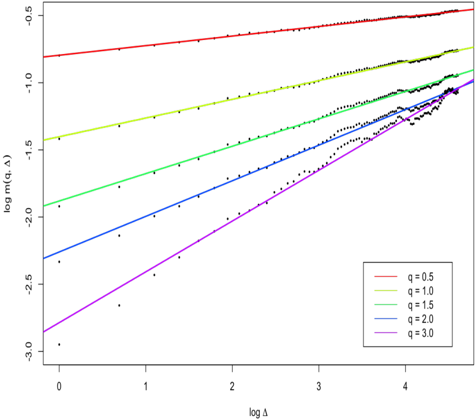
Gatheral-Jaisson-Rosenbaum

Log-volatility behaves as a fractional Brownian Motion with Hurst exponent H of order 0.1 at any reasonable time scale



Gatheral-Jaisson-Rosenbaum

Log-log plot of $m(q, \Delta)$ versus Δ for various q.



Gatheral-Jaisson-Rosenbaum

At-the-money volatility skew $\psi(\tau) = \left| \frac{d}{dk}\Big|_{k=0} \sigma_{BS}(k,\tau) \right|$ is well approximated by a power law function of time to expiry τ

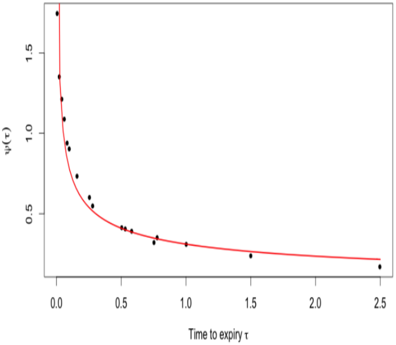


Figure 1.2: The black dots are non-parametric estimates of the S&P ATM volatility skews as of June 20, 2013; the red curve is the power-law fit $\psi(\tau) = A\tau^{-0.4}$.

Fractional volatility process

The observations suggest the following model for instantaneous volatility

$$\sigma_t = \sigma_0 e^{\nu B_t^H},$$

where B^H is a fractional Brownian motion with Hurst exponent H. As stationarity of σ_t is concerned, GJR suggested the model for instantaenous volatility as $\sigma_t = \sigma_0 e^{X_t}$ where

$$dX_t = \alpha(m - X_t)dt + \nu dB_t^H$$

is a fractional Ornstein-Uhlenbeck process. Again, drift term plays no role in large deviation regime.

Review: fractional Brownian motion

A mean-zero Gaussian process B_t^H is called a *fractional Brownian motion* with Hurst exponent $H \in [0,1]$ if its autocovariance function $R(t,s)$, for $t,s>0$, satisfies

$$R(t,s) := \mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

- B^H is self-similar, indeed, $B_{at}^H \overset{d}{=} a^H B_t^H$ for $a>0$
- B^H has stationary increments
- B_t^H is a standard Brownian motion when $H=\frac{1}{2}$
- B_t^H is neither a semimartingale nor Markovian unless $H=\frac{1}{2}$
- B_t^H is Hölder of order β for any $\beta < H$ almost surely

Lognormal fSABR model

Consider the following lognormal fSABR model

$$\frac{dS_t}{S_t} = \alpha_t(\rho dB_t + \bar{\rho} dW_t),$$
$$\alpha_t = \alpha_0 e^{\nu B_t^H},$$

where B_t and W_t are independent Brownian motions, $\bar{\rho} = \sqrt{1-\rho^2}$. B_t^H is a fractional Brownian motion with Hurst exponent H driven by B_t:

$$B_t^H = \int_0^t K_H(t,s)dB_s.$$

K_H is the Molchan-Golosov kernel.

- Goal: to obtain an easy to access expression for the joint density of (S_t, α_t).

Slightly more explicit form

Defining the new variables $X_t = \log S_t$ and $Y_t = \alpha_t$, we may rewrite the lognormal fSABR model in a slightly more explicit form as

$$X_t - X_0 = Y_0 \int_0^t e^{\nu B_s^H}(\rho dB_s + \bar{\rho} dW_s) - \frac{Y_0^2}{2} \int_0^t e^{2\nu B_s^H} ds,$$
$$Y_t = Y_0 e^{\nu B_t^H}.$$

- We derive a bridge representation for the joint density of (X_t, Y_t) in a "Fourier space".

Bridge representation for joint density

The joint density of (X_t, Y_t) has the following bridge representation

$$
\begin{aligned}
&p(t, x_t, y_t | x_0, y_0) \\
= \;& \frac{e^{-\frac{\eta_t^2}{2\nu^2 t^{2H}}}}{y_t\sqrt{2\pi\nu^2 t^{2H}}} \times \frac{1}{2\pi} \times \\
& \int e^{i(x_t - x_0)\xi} \mathbb{E}\left[e^{i\left(-\rho \int_0^t y_0 e^{\nu B_s^H} dB_s + \frac{y_0^2}{2} v_t\right)\xi} e^{-\frac{\bar{\rho}^2 y_0^2 v_t}{2}\xi^2} \,\middle|\, \nu B_t^H = \eta_t \right] d\xi,
\end{aligned}
$$

where $i = \sqrt{-1}$, $v_t = \int_0^t e^{2\nu B_s^H} ds$ and $\eta_t = \log \frac{y_t}{y_0}$.

Bridge representation in uncorrelated case

The bridge representation for the joint density of (X_t, Y_t) reads simpler when $\rho = 0$:

$$p(t, x_t, y_t | x_0, y_0) = \frac{e^{-\frac{\eta_t^2}{2\nu^2 t^{2H}}}}{y_t\sqrt{2\pi\nu^2 t^{2H}}} \times \frac{1}{2\pi}\int e^{i(x_t - x_0)\xi}\mathbb{E}\left[e^{-\frac{1}{2}(\xi - i)\xi y_0^2 v_t}\middle| \nu B_t^H = \eta_t\right] d\xi,$$

where $i = \sqrt{-1}$, $v_t = \int_0^t e^{2\nu B_s^H} ds$ and $\eta_t = \log \frac{y_t}{y_0}$.

McKean kernel

The McKean kernel $p_{\mathbb{H}^2}(t, x_t, y_t | x_0, y_0)$ reads

$$p_{\mathbb{H}^2}(t, x_t, y_t | x_0, y_0) = \frac{\sqrt{2} e^{-t/8}}{(2\pi t)^{3/2}} \int_d^\infty \frac{\xi e^{-\xi^2/2t}}{\sqrt{\cosh \xi - \cosh d}} d\xi,$$

where $d = d(x_t, y_t; x_0, y_0)$ is the geodesic distance from (x_t, y_t) to (x_0, y_0).

- Note that the McKean kernel is a density with respect to the Riemannian volume form $\frac{1}{y_t^2} dx_t dy_t$.
- The bridge representation can be regarded as a generalization of the McKean kernel.
- Indeed, in the case where $H = \frac{1}{2}$, $\nu = 1$ and $\rho = 0$, Ikeda-Matsumoto showed how to recover the McKean kernel.

Expanding around b_s

We expand the conditional expectation in the bridge representation around the deterministic path b_s. Let $\mathbb{E}_{\eta_t}[\cdot] = \mathbb{E}\left[\cdot|\nu B_t^H = \eta_t\right]$. First, define the deterministic path b_s by

$$b_s = \log \mathbb{E}_{\eta_t}\left[e^{2\nu B_s^H}\right].$$

Indeed,

$$\begin{aligned} b_s &= \log \mathbb{E}_{\eta_t}[e^{2\nu B_s^H}] = 2\nu \mathbb{E}_{\eta_t}[B_s^H] + 2\nu^2 \mathrm{var}_{\eta_t}[B_s^H] \\ &= 2R(1,u)\eta_t + 2\nu^2 t^{2H}\left\{u^{2H} - R^2(1,u)\right\}, \end{aligned}$$

where $u = \frac{s}{t}$ and $R(t,s) = \mathbb{E}\left[B_t^H B_s^H\right]$.

- Note that $e^{b_s} = \mathbb{E}_{\eta_t}\left[e^{2\nu B_s^H}\right]$. In other words, e^{b_s} is an unbiased estimator of $e^{2\nu B_s^H}$ conditioned on $\nu B_t^H = \eta_t$.

Now expand the conditional expectation in the bridge representation around the deterministic path b_s as

$$\begin{aligned} & \mathbb{E}_{\eta_t}\left[e^{-\frac{1}{2}(\xi - i)\xi \int_0^t y_0^2 e^{2\nu B_s^H} ds} \right] \\ = \;& e^{-\frac{1}{2}(\xi - i)\xi \int_0^t y_0^2 e^{b_s} ds} \mathbb{E}_{\eta_t}\left[e^{-\frac{1}{2}(\xi - i)\xi \int_0^t y_0^2 \left(e^{2\nu B_s^H} - e^{b_s} \right) ds} \right] \\ \approx \;& e^{-\frac{1}{2}(\xi - i)\xi \int_0^t y_0^2 e^{b_s} ds} \times \{1 + o(1)\}. \end{aligned}$$

Substituting the last expansion into bridge representation we obtain the following expansion (in the Fourier space) in terms of the H_k functions as

$$\begin{aligned} & p(t, x_t, y_t | x_0, y_0) \\ \approx \quad & \frac{1}{y_t \sqrt{2\pi \nu^2 t^{2H}}} e^{-\frac{\eta_t^2}{2\nu^2 t^{2H}}} \times \\ & \frac{1}{2\pi} \int e^{i(x_t - x_0)\xi} \, e^{-\frac{1}{2}(\xi - i)\xi \hat{v}_t} \, \{1 + o(1)\} \, d\xi, \end{aligned}$$

where $\hat{v}_t = \int_0^t y_0^2 e^{b_s} ds$.

Small time asymptotics - uncorrelated

To the lowest order as $t \to 0$, the density p has the following small time asymptotic behaviour

$$p(t, x_t, y_t | x_0, y_0) = \frac{e^{-\frac{\eta_t^2}{2\nu^2 t^{2H}}}}{y_t \sqrt{2\pi\nu^2 t^{2H}}} \frac{e^{-\frac{(x_t - x_0)^2}{2 y_0^2 \hat{v}_t}}}{\sqrt{2\pi y_0^2 \hat{v}_t}} e^{\frac{x_t - x_0}{2}} \{1 + o(1)\},$$

where recall that $\hat{v}_t = \int_0^t e^{b_s} ds$.

Probability density in small time - correlated case

For the correlated case, define the functions C_{RK} and C_{eR} by

$$C_{RK}(\eta) := \int_0^1 e^{R(1,u)\eta} K_H(1,u)du, \quad C_{eR}(\eta) := \int_0^1 e^{2R(1,u)\eta} du.$$

To the lowest order we have

$$\begin{aligned} & p(t, x_t, y_t | x_0, y_0) \\ \approx \quad & \frac{1}{2\pi} \times \frac{1}{y_t\sqrt{\nu^2 t^{2H}}} e^{-\frac{\eta_t^2}{2\nu^2 t^{2H}}} \times \frac{1}{y_0\sqrt{\tilde{v}_t}} e^{-\frac{1}{2y_0^2\tilde{v}_t}\left(x_t - x_0 - \rho y_0 C_{RK}(\eta_t)\frac{\eta_t}{\nu} t^{\frac{1}{2}-H}\right)^2} \end{aligned}$$

where $\tilde{v}_t = t\psi(\eta_t) := \left\{ C_{eR}(\eta_t) - \rho^2 C_{RK}^2(\eta_t) \right\} t.$

Approximate distance function

Rewrite the joint density p as

$$p(t, x_t, y_t | x_0, y_0) \approx \frac{1}{2\pi} \frac{1}{y_t\sqrt{\nu^2 t^{2H}}} \frac{1}{y_0\sqrt{\tilde{v}_t}} e^{-\frac{\tilde{d}^2(x_t, y_t | x_0, y_0)}{2t^{2H}}}$$

where

$$\tilde{d}(x_t, y_t | x_0, y_0) := \frac{\eta_t^2}{\nu^2} + \frac{1}{y_0^2 \psi(\eta_t)} \left(\frac{x_t - x_0}{t^{\frac{1}{2}-H}} - \rho y_0 C_{RK}(\eta_t) \frac{\eta_t}{\nu} \right)^2$$

is regarded as the approximate "distance function".

Convexity of approximate distance function

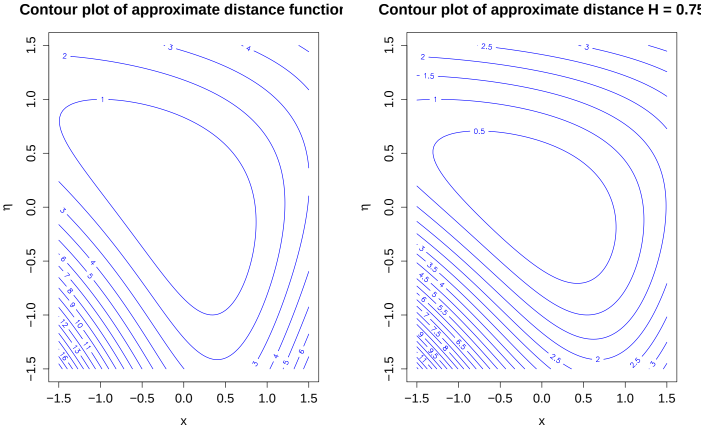


Figure: The contour plots. Parameters $\rho = -0.7$, $\nu = 1$, $y_0 = 1$, $t = 0.5$. $H = 0.75$ on the right; $H = 0.25$, on the left.

Implied volatility approximation by bridge representation

By matching with the Black-Scholes price to the lowest order, we obtain a small time approximation of the implied volatility as follows. Let $\alpha = \frac{1}{2} - H$ and $k = \log \frac{K}{s_0}$.

Implied volatility approximation

$$\sigma_{\mathrm{BS}}^2 \approx \frac{k^2}{T^{2\alpha}} \left\{ \frac{\eta_*^2}{\nu^2} + \frac{1}{y_0^2 \psi(\eta_*)} \left(\frac{k}{T^\alpha} - \rho y_0 C_{RK}(\eta_*) \frac{\eta_*}{\nu} \right)^2 \right\}^{-1}$$

where η_* is the minimizer

$$\eta_* = \operatorname{argmin} \left\{ \eta \in \mathbb{R} : \frac{\eta^2}{\nu^2} + \frac{1}{y_0^2 \psi(\eta)} \left(\frac{k}{T^\alpha} - \rho y_0 C_{RK}(\eta) \frac{\eta}{\nu} \right)^2 \right\}.$$

Note that $\eta_* = \eta_* \left(\frac{k}{T^\alpha} \right)$

Approximate implied volatility plots - 1

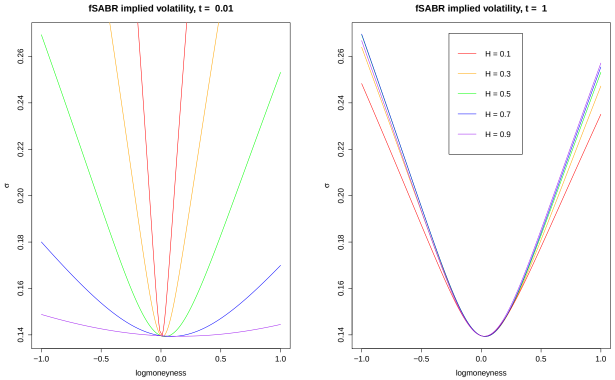


Figure: The implied volatility curves. $t = 0.01$ on the left, $t = 1$ on the right. Parameters are set as $\rho = -0.06867$, $\nu = 0.58$, $\alpha_0 = 0.13927$. $H = 0.1$ in red, $H = 0.3$ in orange, $H = \frac{1}{2}$ in green, $H = 0.7$ in blue, $H = 0.9$ in purple.

Approximate implied volatility plots - 2

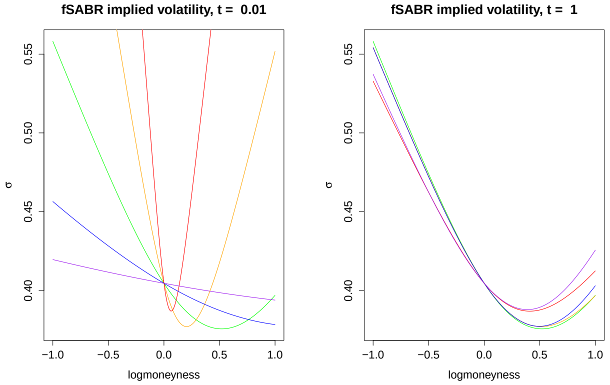


Figure: The implied volatility curves. $t = 0.01$ on the left, $t = 1$ on the right. Parameters are set as $\rho = -0.4$, $\nu = 0.58$, $\alpha_0 = 0.38$. $H = 0.1$ in red, $H = 0.3$ in orange, $H = \frac{1}{2}$ in green, $H = 0.7$ in blue, $H = 0.9$ in purple.

Recovery of SABR formula?

- Q: Does it recover the SABR formula to the lowest order when $H = \frac{1}{2}$?

SABR formula

$$\sigma_{\mathrm{BS}}(k) \approx \frac{-\nu k}{D(\zeta)}, \quad \zeta = -\frac{\nu}{\alpha_0} k,$$

where

$$D(\zeta) = \log \frac{\sqrt{1 - 2\rho\zeta + \zeta^2} + \zeta - \rho}{1 - \rho}.$$

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  A: NO!

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Graphic comparison with SABR formula

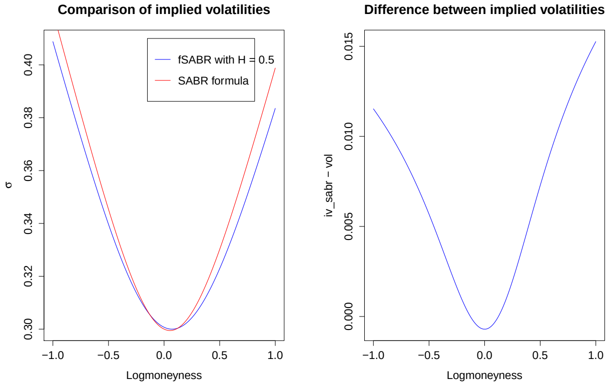

Figure: The implied volatility curves from SABR and fSABR formula. Parameters are set as $\tau = 1$, $\rho = -0.06867$, $\nu = 0.58$, $\alpha_0 = 0.13927$.

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  A: Most-likely-path from bridge representation

Large deviations principle for fSABR

We have as $T \to 0$

$$
\begin{aligned}
&-\log \mathbb{P}\left[X_t = x_t,\, Y_t = y_t \text{ for } t \in [0,T]\right] \\
\approx\quad & \frac{1}{2}\int_0^T \frac{1}{\bar{\rho}^2 y_t^2}\left(\dot{x}_t - \rho y_t b_t\right)^2 dt + \frac{1}{2}\int_0^T b_t^2 dt
\end{aligned}
$$

where $b \in L^2[0,T]$ satisfying

$$
\eta_t = \log y_t - \log y_0 = \nu \int_0^t K_H(t,s) b_s ds
$$

for $t \in [0,T]$.
This should be the rate function for sample path LDP.

Recovery of Freidlin-Wentzell when $H = \frac{1}{2}$

Indeed,

$$b_t = \frac{1}{\nu} K_H^{-1}[\eta](t).$$

When $H = \frac{1}{2}$, K_H^{-1} is simply the usual differential operator, thus

$$b_t = \frac{\dot{\eta}_t}{\nu} = \frac{1}{\nu} \frac{\dot{y}_t}{y_t} .$$

Therefore, the rate function reduces to

$$\begin{aligned} & -\log \mathbb{P}\left[X_t = x_t, Y_t = y_t \text{ for } t \in [0, T]\right] \\ = & \frac{1}{2} \int_0^T \frac{1}{\bar{\rho}^2 y_t^2} \left(\dot{x}_t - \rho y_t \frac{\dot{\eta}_t}{\nu} \right)^2 dt + \frac{1}{2} \int_0^T \left(\frac{\dot{\eta}_t}{\nu} \right)^2 dt \\ = & \frac{1}{2} \int_0^T \frac{1}{\bar{\rho}^2 \nu^2 y_t^2} \left(\nu^2 \dot{x}_t^2 - 2\rho\nu \dot{x}_t \dot{y}_t + \dot{y}_t^2 \right) dt \end{aligned}$$

which recovers the classical large deviations principle of Freidlin-Wentzell

Implied volatility approximation by LDP

Again, by matching with the Black-Scholes price, we obtain

fSABR formula

$$\sigma_{\mathrm{BS}}^2 \approx \frac{k^2}{T} \left(\int_0^T \frac{1}{\bar{\rho}^2 {y_t^*}^2} \left(\dot{x}_t^* - \rho y_t^* b_t^*\right)^2 + {b_t^*}^2 dt \right)^{-1},$$

where (x^*, b^*) is the minimizer

$$(x^*, b^*) = \operatorname{argmin} \left\{ \dot{x}, b \in L^2[0,T] : \int_0^T \frac{1}{\bar{\rho}^2 y_t^2} \left(\dot{x}_t - \rho y_t b_t\right)^2 + b_t^2 dt \right\}$$

with $x_T = k$ and y_t^* is given by, for $t \in [0,T]$,

$$\log y_t^* - \log y_0 = \nu \int_0^t K_H(t,s) b_s^* ds$$

This recovers the SABR formula when $H = \frac{1}{2}$!

Pricing of Target Volatility Option in fSABR

Target volatility option

Target Volatility Option (TVO)

- is a type of derivative instrument that explicitly depends on the evolution of an underlying asset as well as its realized volatility
- allows one to set a target volatility parameter that determines the leverage of an otherwise price dependent payoff
- is an option whose multiplicative leverage factor is the ratio of the target volatility to the realized volatility of the underlying asset at maturity

Target volatility call

A TV call at expiry pays off

$$\frac{\bar{\sigma}}{\sqrt{\frac{1}{T}\int_0^T \sigma_t^2 dt}}(S_T - K)^+ = \frac{\bar{\sigma}\sqrt{T}K}{\sqrt{\int_0^T Y_t^2 dt}}\left(e^{X_T} - 1\right)^+,$$

where $\bar{\sigma}$ is the (preassigned) *target volatility* level.

- Apparently, if at expiry the realized volatility is higher (lower) than the target volatility, the payoff is scale down (up) by the ratio between target volatility and realized volatility.

We will temporarily ignore the factor $\bar{\sigma}\sqrt{T}K$ hereafter for simplicity.

Normalized Black-Scholes function

The normalized Black-Scholes function C:

$$C(x, w) = e^x N(d_1) - N(d_2)$$

where $d_1 = \frac{x}{\sqrt{w}} + \frac{\sqrt{w}}{2}$ and $d_2 = d_1 - \sqrt{w}$.

- C satisfies the (forward) Black-Scholes PDE

$$C_w = \frac{1}{2} C_{xx} - \frac{1}{2} C_x$$

with initial condition $C(x, 0) = (e^x - 1)^+$.

For any $t \in [0, T]$, define

$$w_t := \int_0^t Y_s^2 ds \quad \text{(total variance up to time } t\text{)}$$

$$\hat{w}_t := \mathbb{E}_t \int_t^T Y_s^2 ds \quad \text{(expected total variance from } t \text{ to } T\text{)}$$

$$M_t := \mathbb{E}_t \int_0^T Y_s^2 ds.$$

Note that M_t is a martingale and $M_t = w_t + \hat{w}_t$.

A decomposition formula for TV call

By applying Itô's formula to the process $\frac{1}{\sqrt{M_t}} C(X_t, \hat{w}_t)$, we obtain for $t \in [0, T]$

$$
\begin{aligned}
& \frac{1}{\sqrt{w_T}} \left(e^{X_T} - 1 \right)^+ \\
= \; & \frac{1}{\sqrt{M_t}} C(X_t, \hat{w}_t) + \int_t^T \frac{C_x}{\sqrt{M_s}} \frac{dS_s}{S_s} + \int_t^T \frac{C_w}{\sqrt{M_s}} dM_s \\
+ \; & \int_t^T \left(-\frac{C_x}{2(\sqrt{M_s})^3} + \frac{C_{xw}}{\sqrt{M_s}} \right) d\langle M, X \rangle_s \\
+ \; & \int_t^T \left(-\frac{C_w}{2(\sqrt{M_s})^3} + \frac{3C}{8(\sqrt{M_s})^5} + \frac{C_{ww}}{2\sqrt{M_s}} \right) d\langle M \rangle_s .
\end{aligned}
$$

The formula suggests a model independent theoretical replicating strategy for TV call, assuming the availability of all variance swaps and swaptions.

A decomposition formula for TV call

Taking conditional expectation of the last equation on both sides yields

$$
\begin{aligned}
& \mathbb{E}_t\left[\frac{1}{\sqrt{w_T}}\left(e^{X_T}-1\right)^+\right] \\
= \quad & \frac{1}{\sqrt{M_t}} C(X_t, \hat{w}_t) \\
+ \quad & \mathbb{E}_t\left[\int_t^T \left(-\frac{C_x}{2(\sqrt{M_s})^3} + \frac{C_{xw}}{\sqrt{M_s}}\right) d\langle M, X\rangle_s\right] \\
+ \quad & \mathbb{E}_t\left[\int_t^T \left(-\frac{C_w}{2(\sqrt{M_s})^3} + \frac{3C}{8(\sqrt{M_s})^5} + \frac{C_{ww}}{2\sqrt{M_s}}\right) d\langle M\rangle_s\right].
\end{aligned}
$$

- If the driving Brownian motions are uncorrelated, the second term on the right hand side vanishes.

Approximation of TV call - zeroth order

As $t \to T$, by dropping the last two terms we obtain

$$\mathbb{E}_t\left[\frac{1}{\sqrt{w_T}}\left(e^{X_T}-1\right)^+\right] \approx \frac{1}{\sqrt{M_t}} C(X_t, \hat{w}_t)$$

- The approximation is exact in the deterministic volatility case.
- In words, to zeroth order at time t, the price of a TV call struck K with expiry T is given by the price of a vanilla call with total variance given by the variance swap between t and T, rescaled by the quantity of the sum of the realized variance from 0 to t and the variance swap between t and T.
- Notice that the zeroth order approximation is independent of ρ. In fact, it is model independent, assuming variance swap is a market observable.

Martingale representation for M_t

Assuming fSABR, by applying the Clark-Ocone formula, M_t has following martingale representation

$$\begin{aligned} M_t &= M_0 + \int_0^t 2\nu \int_s^T \mathbb{E}\left[Y_r^2 \middle| \mathcal{F}_s^B \right] K(r,s) dr dB_s \\ &= M_0 + 2\nu Y_0^2 \int_0^t \int_s^T \mathbb{E}\left[e^{2\nu B_r^H} \middle| \mathcal{F}_s^B \right] K(r,s) dr dB_s, \end{aligned}$$

where

$$\mathbb{E}\left[Y_r^2 \middle| \mathcal{F}_t^B \right] = Y_0^2 e^{2\nu m(r|t) + 2\nu^2 v(r|t)}$$

$$m(r|t) = \frac{B_t}{t} \int_0^t K(r,s) ds,$$

$$v(r|t) = r^{2H} - \frac{1}{t} \left(\int_0^t K(r,s) ds \right)^2 .$$

Thus,

$$d\langle M\rangle_t = 4\nu^2\left(\int_t^T \mathbb{E}\left[\left. Y_r^2\right| \mathcal{F}_t^B\right] K(r,t)dr\right)^2 dt,$$

$$d\langle X, M\rangle_t = 2\nu\rho\left(Y_t\int_t^T \mathbb{E}\left[\left. Y_r^2\right| \mathcal{F}_t^B\right] K(r,t)dr\right) dt.$$

It follows that

$$\langle M\rangle_t = 4\nu^2\int_0^t\left(\int_s^T Y_0^2 e^{2\nu m(r|s)+2\nu^2 v(r|s)}K(r,s)dr\right)^2 ds,$$

$$\langle X, M\rangle_t = 2\nu\rho\int_0^t\int_s^T Y_s Y_0^2 e^{2\nu m(r|s)+2\nu^2 v(r|s)}K(r,s)drds.$$

- Note that $m(\cdot|s)$ is a Gaussian process; whereas $v(\cdot|\cdot)$ is deterministic.

Approximation of TV call - first order

As $t \to T$, we have

$$
\begin{aligned}
& \mathbb{E}_t\left[\frac{1}{\sqrt{w_T}}\left(e^{X_T}-1\right)^+\right] \\
\approx \quad & \frac{C}{\sqrt{M_t}} + \left(-\frac{C_x}{2(\sqrt{M_t})^3} + \frac{C_{xw}}{\sqrt{M_t}}\right)\mathbb{E}_t\left[\int_t^T d\langle M, X\rangle_s\right] \\
& + \left(-\frac{C_w}{2(\sqrt{M_t})^3} + \frac{3C}{8(\sqrt{M_t})^5} + \frac{C_{ww}}{2\sqrt{M_t}}\right)\mathbb{E}_t\left[\int_t^T d\langle M\rangle_s\right],
\end{aligned}
$$

where C and all its partial derivatives are evaluated at $(X_t, \hat{w}_t)$.

Approximation of TV call at time $t = 0$

In particular, at $t = 0$ the approximation simplifies slightly as

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{\sqrt{w_T}}\left(e^{X_T}-1\right)^+\right] \\
\approx\ & \frac{C}{\sqrt{M_0}} + \left(-\frac{C_x}{2(\sqrt{M_0})^3} + \frac{C_{xw}}{\sqrt{M_0}}\right)\mathbb{E}\left[\langle M, X\rangle_T\right] \\
+\ & \left(-\frac{C_w}{2(\sqrt{M_0})^3} + \frac{3C}{8(\sqrt{M_0})^5} + \frac{C_{ww}}{2\sqrt{M_0}}\right)\mathbb{E}\left[\langle M\rangle_T\right],
\end{aligned}$$

where C and all its partial derivatives are evaluated at (X_0, M_0).

Conclusion

- We show a bridge representation for the joint density of the lognormal SABR model.
- Small time asymptotics to the lowest order are presented for option price and implied volatility.
- We show a heuristic derivation of large deviations principle which recovers the classical Freidlin-Wentzell large deviations principle when $H = \frac{1}{2}$.
- We obtain a decomposition formula for TV calls which suggests a theoretical model independent replicating strategy.
- Approximations of TV call price are obtained by "freezing the coefficient".

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THANK YOU FOR YOUR ATTENTION.